Conformal maps: distortion theory

Karl Schwarz (1843-1921)

Schwarz lemma: $f:\mathbb{D}\to\mathbb{D}$, $f(0)=0 \Rightarrow$ 1) $|f(z)|\le|z|$ 2) $|f'(0)|\le 1$ Equality $\iff f(z)=e^{i\theta}z,\ \theta\in\mathbb{R}$.
analytic

Pf. Apply maximum principle to $\varphi(z)=\frac{f(z)}{z}$ ∎

A corollary: self-map of $\mathbb{D}$. $Aut(\mathbb{D}):=\{\varphi:\mathbb{D}\to\mathbb{D}$ - conformal$\}$
$Aut(\mathbb{D})=\{e^{i\theta}\frac{z-z_0}{1-\bar z_0 z},\ z_0\in\mathbb{D},\ e^{i\theta}\in S^1\}$.

Hyperbolic metric:

$$\rho(z_1,z_2)=\inf_{\gamma \text{ joining } z_1,z_2}\int_\gamma\frac{|dz|}{1-|z|^2}=\log\left(\frac{1+\left|\frac{z_2-z_1}{1-\bar z_1 z_2}\right|}{1-\left|\frac{z_2-z_1}{1-\bar z_1 z_2}\right|}\right)\qquad \tanh\rho(z_1,z_2)=\left|\frac{z_2-z_1}{1-\bar z_1 z_2}\right|$$ - pseudohyperbolic metric.

Easy to check: invariant under $Aut(\mathbb{D})$:
$\rho(\varphi(z_1),\varphi(z_2))=\rho(z_1,z_2)$.
Thus, if $f:\mathbb{D}\to\Omega$ - conformal, then $\rho_\Omega(z_1,z_2):=\rho(f^{-1}(z_1),f^{-1}(z_2))$ is well-defined.
(if $g:\mathbb{D}\to\Omega$ - conformal, then $g\circ f^{-1}\in Aut(\mathbb{D})$, so $\rho(f^{-1}(z_1),f^{-1}(z_2))=\rho(g^{-1}(z_1),g^{-1}(z_2))$.
Hyperbolic metric in $\mathbb{H}$: $\frac{|dz|}{y}$.

Schwarz Lemma (invariant form). $f:\Omega_1\to\Omega_2$ - analytic, then
1) $\forall z_1,z_2$: $\rho_{\Omega_2}(f(z_1),f(z_2))\le\rho_{\Omega_1}(z_1,z_2)$
2) Equality for some z_1,z_2 $\iff f:\Omega_1\to\Omega_2$ - conformal.
Pf. Just map everything back to disk by Riemann map ∎

Corollary. $f:\mathbb{D}\to\Omega$ - conformal, then $\mathrm{dist}(f(z),\partial\Omega)\le|f'(z)|(1-|z|^2)$
Pf. Let $g(w):=f^{-1}(f(z)+w\,\mathrm{dist}(f(z),\partial\Omega))$ well-defined, maps $\mathbb{D}\to\mathbb{D}$, $0\to z$.
$h(w):=\frac{g(w)-z}{1-\bar z g(w)}$. $h:\mathbb{D}\to\mathbb{D}$, $h(0)=0$. So $|h'(0)|\le 1$. But
$h'(0)=\frac{\mathrm{dist}(f(z),\partial\Omega)}{|f'(z)|(1-|z|^2)}$ ∎

Adolf Hurwitz (1859-1919)

Thm (Hurwitz). (f_n): conformal in Ω, $f_n \to f$ uniformly on compacts. $\Rightarrow$ either $f \equiv$ const in Ω or f is conformal in Ω.

Pf. 1) f is analytic, conformal.

2) Let $f(z_1) = f(z_2)$. $g_n(z) := f_n(z) - f_n(z_2)$, $g(z) := \lim_{n\to\infty} g_n(z)$. If $g \not\equiv 0$, then $\exists\, \gamma$ - surrounding z_1 but not z_2 such that $g(z) \neq 0$ on γ. Let $\delta := \min|g(z)|$. $\exists N: n > N: |g_n - g| < \delta$ on γ. g has zero inside γ (z_1), so is g_n, by Rouche Theorem. Thus $g_n(w) = g_n(z_2)$ for some w inside γ - contradicts 1-1! ∎

Paul Montel(1876-1975)

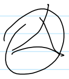
Def. Normal family of functions is pre-compact in $A(\Omega)$ (i.e. $\forall (f_n) \exists (f_{n_k})$: $f_{n_k} \to f$ uniformly on compacts).

Easy Thm. Any locally bounded family is precompact in $A(\Omega)$.

Pf. Arzela - Ascoli + Cauchy ∎

Thm (Montel). Let $\mathcal{F} \subset A(\Omega)$ omits two points, i.e. $\exists\, a \neq b \in \mathbb{C}: \forall f \in \mathcal{F}\ f^{-1}(a) = \emptyset$, $f^{-1}(b) = \emptyset$. Then $\mathcal{F}$ is normal.

Pf. $\exists$ covering $\pi: \mathbb{D} \to \mathbb{C} \setminus \{0,1\}$ - modular function.



Can define hyperbolic distance.

$\forall f_n$: $f_n(0)$ converges (can always select converging subsequence in $\hat{\mathbb{C}}$).

$f_n(0) \to z \in \hat{\mathbb{C}}$. If $z \in \{0, 1, \infty\}$, by Schwarz lemma, $f_n(w) \to z\ \forall w$.

Otherwise, lift f_n to the cover; $g_n: (\mathbb{D}, 0) \to (\mathbb{D}, \xi_n)$. Bounded, so $\exists\, g_{n_k}$ - converging, so is $f_{n_k} = \pi \circ g_{n_k}$ ∎

$$\begin{array}{ccc} \mathbb{D} & \xleftarrow{g_n} & \\ \downarrow \pi & & \\ \mathbb{C}\setminus\{0,1,\infty\} & \xleftarrow{f_n} & \mathbb{D} \end{array}$$

Normalization:

$(S) = \{f$ - conformal in $\mathbb{D}$, $f(0) = 0$, $f'(0) = 1\} = \{f(z) = z + \sum_{k\geq 2} a_n z^n\}$.

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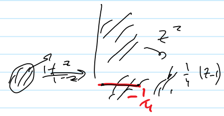
$(\Sigma) = \{ f$ – conf in $\mathbb{D}^*$, $f(z) = z + b_0 + \frac{b_{-1}}{z} + \ldots$ near $\infty \}$.

Connection: $f(z) \in (S) \iff 1/f(\frac{1}{z}) \in (\Sigma)$

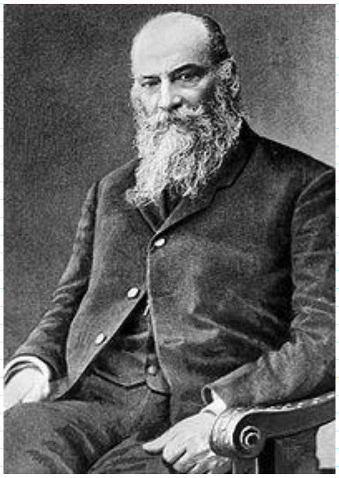
$g(z) \in (\Sigma)$, $c \notin g(\mathbb{D}^*) \Leftarrow \frac{1}{g(1/z) - c} \in (S)$

Examples (important!): 1) Köbe function $k(z) = \frac{z}{(1-z)^2} = z + 2z^2 + 3z^3 + \ldots$



(since $k(z) = \frac{1}{4}\left[\left(\frac{1+z}{1-z} \right)^2 - 1 \right]$

Remark $k_\alpha(z) := \frac{1}{2\alpha}\left[\left(\frac{1+z}{1-z} \right)^\alpha - 1 \right] \in (S)$ for $\alpha \in (0, 2]$.



Nikolai Zhukovsky(1847-1921)

2) Zhukovsky transform:

$f(z) = z + \frac{1}{z} \in (\Sigma)$, $f : \mathbb{D}_* \to \mathbb{C} \setminus [-2, 2]$.

Transforms preserving (S)

1) n th root transform: $f \to z \sqrt[n]{\frac{f(z^n)}{z^n}}$ – n-folt symmetrization $\in (S)$.

$g(z_1) = g(z_2) \Rightarrow g^n(z_1) = g^n(z_2) \Rightarrow z_1^n = z_2^n$. But $g(z_1) = z_1 \sqrt{\ldots} \ne z_2 \sqrt{\ldots} = g(z_2)$.

2) Köbe transform: $\tau \in Aut(\mathbb{D})$, $f_\tau := \frac{f \circ \tau(z) - f \circ \tau(0)}{(f \circ \tau)'(0)}$

In particular, for $\tau(z) = e^{i\theta} z$, $f_\tau = e^{-i\theta} f(e^{i\theta} z)$

3) Symmetry: $f \to \overline{f(\bar{z})}$.

Thm (Köbe). (S) is compact in $A(\mathbb{D})$.

Pf. Closed by Hurwitz (since $f_n \to f \Rightarrow f_n'(0) \to f'(0)$).

Normality: By Schwarz: $\forall f \in (S)\ \exists\ a(f), b(f) \notin f(\mathbb{D})$, $|a(f)| = 1$, $|b(f)| = 2$.

Take (f_n). $A_n := \frac{(z - a_n)}{b_n - a_n}$ $A_n \circ f_n : \mathbb{D} \to \mathbb{C} \setminus \{0, 1, \infty\}$. So, by Montel, $\exists$ conv. subsequence. $A_{n_k} \circ f_{n_k}$. By passing to further subsequence, $a_{n_k} \to a$, $b_{n_k} \to b$ $(a \neq b)$ since $|a| = 1$, $|b| = 2$. So $A_{n_k} \to A$, so $A_{n_k} \circ f_{n_k} \to A \circ f \Rightarrow f_{n_k} \to f$

Ludwig Bieberbach(1886-1982)

Paul Koebe (1882-1945)

Harder Theory : Koebe-Bieberbach.

Lem 1: (Gronwall Area Thm).
$F \in (\Sigma) \Rightarrow \sum_{n=1} n|b_{-n}|^2 \leq 1$.

Pf. Let $R \geq 1$, $\Omega_R := \mathbb{C} \setminus F(R\mathbb{D}^*)$.
$\gamma_R := F(RS^1)$ (well-defined for $R>1$).
By Green's formula, $\text{Area}(\Omega_R) = -\frac{i}{2}\int_{\gamma_R} w\,d\bar{w} = -\frac{i}{2}\int_0^{2\pi} F(Re^{i\theta})\overline{\frac{\partial F}{\partial \theta}(Re^{i\theta})}\,d\theta$.
$F(Re^{i\theta}) = Re^{i\theta} + \sum b_{-n} R^{-n} e^{-in\theta}$
$\frac{\partial F}{\partial \theta} = i\,(Re^{i\theta} - \sum n b_{-n} R^{-n} e^{-in\theta})$.
Since $\int f\bar{g} = \pi \sum a_n \bar{b}_n$ - Parceval, $\text{Area}(\Omega_R) = \pi\left(R^2 - \sum n \frac{|b_{-n}|^2}{R^{2n}}\right)$. Let $R \to 1$. ∎

Remark. Equality $\Leftrightarrow A(R) \to 0 \Leftrightarrow \text{Area}(\Omega_1) = 0$.

Corollary. $|b_{-1}| \leq 1$, ($= 1 \Leftrightarrow F$ is a shift/rotation of Zhukovsky function). ∎

Thm. (Bieberbach). $|a_2| \leq 2$ for $f \in (S)$

Remark. de Branges, 1984, finally proved that $|a_n| \leq n$, so Köbe is extremal.
Pf. Let $g(z) := z\sqrt{\frac{f(z^2)}{z^2}} = 1 + \frac{a_2}{2} z^2 + \ldots \in (S)$
$(\Sigma) \ni G(z) = \frac{1}{g(\frac{1}{z})} = z + \frac{a_2}{2} \cdot \frac{1}{z} + \ldots$ so $|a_2| \leq 2$ ∎

Important Köbe Thm. $f \in (S) \Rightarrow \frac{1}{4}\mathbb{D} \subset f(\mathbb{D})$.
Pf. $w \notin f(\mathbb{D}) \Rightarrow g(z) := \frac{wf(z)}{w - f(z)} = z + (a_2 + \frac{1}{w})z^2 + \ldots \in (S)$
By Bieberbach, $|a_2 + \frac{1}{w}| \leq 2$, $|a_2| \leq 2 \Rightarrow |\frac{1}{w}| \leq 4 \Rightarrow |w| \geq \frac{1}{4}$ ∎

Remark. Exact: Köbe function.

Corollary 1 $f: \mathbb{D} \to \Omega$ – conformal $\Rightarrow \frac{1}{4} |f'(z)|(1-|z|^2) \le \operatorname{dist}(f(z), \partial\Omega) \le |f'(z)|(1-|z|^2)$.

Pf. RHS – already derived from Schwarz.

LHS: for $z_0 \in \mathbb{D}$, $g(z) := \dfrac{f\left(\frac{z+z_0}{1+\bar{z}_0 z}\right) - f(z_0)}{|f'(z_0)|(1-|z_0|^2)} \in (S)$, so $w \notin f(\mathbb{D}) \Rightarrow \dfrac{|w-f(z_0)|}{|f'(z_0)|(1-|z_0|^2)} \ge \frac{1}{4}$

Corollary 2 $\varphi: \Omega_1 \to \Omega_2$ – conformal $\Rightarrow \frac{1}{4} \le \dfrac{\operatorname{dist}(\varphi(z), \partial\Omega_2)}{|\varphi'(z)| \operatorname{dist}(z, \partial\Omega_1)} \le 4$

Def. Quasi-hyperbolic metric:

$$Q_\Omega(z_1, z_2) = \inf_{\gamma_{z_1,z_2}} \int \frac{|dw|}{\operatorname{dist}(w, \partial\Omega)}.$$

Corollary 3 $Q_\Omega(z_1,z_2) \le \rho_\Omega(z_1,z_2) \le 4 Q_\Omega(z_1,z_2)$.

Witney squares are almost conf. invariant!

Thm (Bieberbach inequality) $f \in (S)$, $|z| = r$. Then

$$\left| \frac{z f''(z)}{f'(z)} - \frac{2r^2}{1-r^2} \right| \le \frac{4r}{1-r^2} \qquad \text{(Exact: köbe function).}$$

Pf. $T(\zeta) := \dfrac{\zeta + z}{1+\bar{z}\zeta}$ $\quad g(\zeta) := f_T(\zeta) = \zeta + a_2\zeta^2 + \dots$, $a_2 = \frac{1}{2}\left[\left(1-|z|^2\right)\frac{f''(z)}{f'(z)} - 2\bar{z}\right]$

$|a_2| \le 2$. Multiply by $\dfrac{2|z|}{1-|z|^2}$.

Observe $\operatorname{Re} \dfrac{z g'(z)}{|z|} = \dfrac{\partial \operatorname{Re} g}{\partial r}$, $\quad \operatorname{Im} \dfrac{z g'(z)}{|z|} = \dfrac{\partial \operatorname{Im} g}{\partial r}$.

$\dfrac{f''}{f'} = (\log f')'$; $\quad \dfrac{\partial \log|f'|}{\partial r} = \dfrac{\partial \operatorname{Re} \log f'}{\partial r} = \operatorname{Re} \dfrac{z(\log f')'}{|z|}$

so we get, $\dfrac{2r-4}{1-r^2} \le \dfrac{\partial}{\partial r} \log|f'(z)| \le \dfrac{2r+4}{1-r^2}$. Integrate $\int_0^r$ to get the result;

Thm (Distortion Thm) $\dfrac{1-r}{(1+r)^3} \le |f'(z)| \le \dfrac{1+r}{(1-r)^3}$.

If we use Im part, we get

$-\frac{4}{1-r^2} \le \frac{\partial}{\partial r} \arg f'(re^{i\theta}) \le \frac{4}{1-r^2} \Rightarrow$ **Thm (Rotation Thm)** $|\arg f'(z)| \le 2\log\frac{1+r}{1-r}$.

Impresice! Presice: $|\arg f'(re^{i\theta})| \le \begin{cases} 4\arcsin r, & r < \frac{1}{\sqrt{2}} \\ \pi + \log\left(\frac{r^2}{1-r^2}\right), & r \ge \frac{1}{\sqrt{2}} \end{cases}$ – using Löwner chains

Moreover, can get

Thm (Growth Thm) $\dfrac{r}{(1+r)^2} \le |f(z)| \le \dfrac{r}{(1-r)^2}$

Pf.

Integrate again: $|f(z)| \le \int |f'(se^{i\theta})|\, ds \le \int_0^r \frac{1+s}{(1-s)^3} ds = \frac{r}{(1-r)^2}$.

Lower bound: let z_0 be the point with $\{|z| = r\}$ with the smallest $|f(z_0)|$.

Then the segment $[0, f(z_0)] \subset f(r\mathbb{D})$, so $\gamma := f^{-1}([0, f(z_0)]) \subset r\mathbb{D}$; joins 0 to z_0.

$\therefore$ $|f(z_0)| = \int_{f(\gamma)} |dw| = \int_\gamma |f'(z)|\,|dz| \geq \int_\gamma \frac{1-|z|}{(1+|z|)^3}|dz| \geq \int_0^r \frac{1-|z|}{(1+|z|)^3}\, d|z| = \frac{r}{(1+r)^2}$ ∎

Thm $\frac{1-r}{r(1+r)} \leq \frac{|f'(z)|}{|f(z)|} \leq \frac{1+r}{r(1-r)}$

Pf. Use $\tau(\zeta) = \frac{\zeta+z}{1+\bar{z}\zeta}$, & $f_\tau(\zeta)$, use at $\zeta = -z_0$, to see that

$\left|z\frac{f'(z)}{f(z)}\right| = \frac{r}{(1-r^2)|f_\tau(-z)|}$ and use growth Thm ∎

André Bloch (1893 – 1948)

Bloch norm

Def $\|b\|_{\mathcal{B}} = \sup(1-|z|^2)|b'(z)|$ ~ Bloch norm. (Actually: seminorm, $\|\text{const}\|_{\mathcal{B}} = 0$).

Intuitively: Lipshitz from hyperbolic to euclidean

Property: $\tau \in \mathrm{Aut}(\mathbb{D})$, $b \in \mathcal{B} \Rightarrow \|b\|_{\mathcal{B}} = \|b \circ \tau\|_{\mathcal{B}}$.

Thm. $f \in (S) \Rightarrow \|\log f'\|_{\mathcal{B}} \leq 6$.

Pf. By Bieberbach: $r\left|\frac{f''(re^{i\theta})}{f'(re^{i\theta})}\right| \leq \frac{2r^2}{1-r^2} + \frac{4r}{1-r^2} \leq \frac{6r}{1-r^2}$ ∎

Remark. Univalence criterium (Nehari): $f \in A(\mathbb{D})$, $f' \neq 0$, $\|\log f'\|_{\mathcal{B}} \leq 1 \Rightarrow f$ is univalent

Corollary. $\exists C$: $\left|\frac{f'(z_1)}{f'(z_2)}\right| \leq C\rho(z_1, z_2)$ for $f \in A(\mathbb{D})$, f - conf.

(S) and Hardy spaces. Def. Hardy space $H^p(\mathbb{D}) = \{f \in A(\mathbb{D}) : \sup \int_0^{2\pi} |f(re^{i\theta})|\, d\theta < \infty\}$

Prawitz Thm. $(S) \subset H^p$ $\forall$ $0 < p < 1/2$

(S) and Hardy spaces. <u>Def.</u> Hardy space $H^p(\mathbb{D}) = \{f \in A(\mathbb{D}) : \sup \int_0^{2\pi} |f(re^{i\theta})|^p d\theta < \infty\}$.

Pravitz Thm $(S) \subset H^p \quad \forall\ 0 < p < 1/2$

Follows from

Pravitz lemma For $f \in (S)$, $\frac{1}{2\pi}\int |f(r\zeta)|^p |d\zeta| \le p\int_0^r \frac{M(\rho)^p}{\rho} d\rho$, where

$M(\rho) = \max_{|z|=\rho} |f(z)|$.

<u>Pf</u> (of Pravitz Lemma)

$$\frac{d}{dr}\int_0^{2\pi} |f(re^{i\theta})|^p d\theta = \int \frac{\partial |f(re^{i\theta})|^p}{\partial r} d\theta = \int p|f(re^{i\theta})|^p \frac{\partial \log|f(re^{i\theta})|}{\partial r} d\theta$$

$$\underset{\text{Cauchy Riemann for } \log f(re^{i\theta})}{=} \int p|f(re^{i\theta})|^p \frac{\partial \arg f(re^{i\theta})}{r\,\partial\theta} d\theta \underset{w=f(z)}{=} \frac{1}{r}\int_{f(rS^1)} p|w|^p d\arg w$$

But $f(rS^1) \subset M(r)\mathbb{D}$, so,

$$\int_{f(rS^1)} |w|^p d\arg w \le (M(r))^p \cdot 2\pi, \text{ since}$$

$$\int_{M(r)S^1} |w|^p d\arg w - \int_{f(rS^1)} |w|^p d\arg w = p\iint_\Omega |w|^{p-2} dx\,dy$$

$M(r)S^1$

$\Omega := M(r)\mathbb{D} \setminus f(rS^1)$

Integrate $\frac{1}{2\pi}\int |f(re^{i\theta})|^p d\theta = \frac{1}{\pi}\int_0^r \frac{\partial}{\partial\rho}\int |f(\rho e^{i\theta})|^p d\theta\, d\rho$

<u>Proof (of Pravitz theorem).</u>

By Growth Thm, $M(\rho) \le \frac{\rho}{(1-\rho)^2}$

For $p < \frac{1}{2}$ $\quad \int_0^{2\pi} |f(re^{i\theta})|^p d\theta \le 2\pi p \int_0^r \frac{M(\rho)^p}{\rho} \lesssim 2\pi p \int_0^1 \rho^{-p}(1-\rho)^{-2p} d\rho$

converges! ∎